Atomic Cut Introduction by Resolution: Proof Structuring and Compression

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Abstract

The careful introduction of cut inferences can be used to structure and possibly compress formal sequent calculus proofs. This paper presents CIRes, an algorithm for the introduction of atomic cuts based on various modifications and improvements of the CERes method, which was originally devised for efficient cut-elimination. It is also demonstrated that CIRes is capable of compressing proofs, and the amount of compression is shown to be exponential in the length of proofs.

1 Introduction

It is well-known that eliminating cuts frequently increases the size and length of proofs. In the worst case, cut-elimination can produce non-elementarily larger and longer proofs [16, 15]. Given this fact, it is natural to attempt to a desire to devise methods that could introduce cuts and compress sequent calculus[1] proofs. However, this has been a notoriously difficult task. Indeed, the problem of answering, given a proof φ and a number l such that $l \leq length(\varphi)$, whether there is a proof ψ of the same theorem and such that $length(\psi) < l$ is known to be undecidable [6]. Nevertheless, a lower bound for compressibility based on specific cut-introduction methods that are inverse of reductive cut-elimination methods is known [10][2], and some ad-hoc methods to introduce cuts of restricted forms have been proposed. They are based on techniques from automated theorem proving, such as conflict-driven formula learning [9], and from logic programming, such as tabling [14, 13].

Besides compression, cut-introduction can also be used for structuring and extracting interesting concepts from proofs. In [8], for example, it is shown that many translation and pre-processing techniques of automated deduction can be seen as introduction of cuts, from a proof-theoretical point of view. Furthermore, in the formalization of mathematical proofs, lemmas correspond to cuts, and hence the automatic introduction of cuts is, in a formal level, the automatic discovery of lemmas that are potentially useful for structuring mathematical knowledge. Naturally, the use of cut-introduction techniques could in principle also be applied to the structuring of knowledge in other fields of Science, as argued in [18, 19].

This paper presents a new method for the introduction of atomic cuts: CIRes. The method is described in a simplified and self-contained manner[3] in Section 2, and a proof that it is able to provide an exponential compression in the length of proofs is given in Section 3.

[1]The sequent calculus used in this paper is the purely multiplicative version of **LK**. Its Inference rules are shown in Appendix A .

[2]A cut-introduction method g is inverse of a reductive cut-elimination method if and only if, for any cut-free proof φ, the proof with cuts $g(\varphi)$ rewrites to φ according to the rewriting rules in Appendix B .

[3]A more general and technically more detailed definition of the CIRes method is available in [18].

2 The CIRes Method

Curiously, even though CIRes aims at introducing cuts, it makes use of the CERes method [3], which was originally developed for efficient cut-elimination [4]. The essential idea behind CIRes is based on two simple observations:

- In a naive attempt to introduce cuts by applying the proof rewriting rules of reductive cut-elimination methods (i.e. gentzen-style cut-elimination methods) shown in Appendix B in an inverse direction, the first step, which is the introduction of atomic cuts on the top of cut-free proofs, is trivial. However, pushing the cuts down (by applying inverse rank reduction rules), combining the cuts to make more complex cuts (i.e. increasing the grade), and exploiting redundancies in the form of contractions is highly non-trivial.

- If applied to a proof ψ containing only atomic cuts in the top, CERes outputs a proof ψ' containing atomic cuts in the bottom. This is so because ψ' is constructed by composing several cut-free parts of ψ, called projections, on top of a resolution refutation. The refutation occurs in the bottom of ψ', and hence the atomic cuts of ψ', which are nothing else but the resolution inferences of the refutation, also occur in the bottom of ψ'.

The CIRes method exploits these observations in a conceptually simple way: it trivially adds atomic cuts to every leaf of the cut-free proof; and then it applies CERes to push these cuts down.

Compression can be achieved mainly due to two ways by which CERes is able to reduce or avoid redundancies:

- It is possible that the refutation uses only some clauses of the clause set of ψ. The effect is that large parts of ψ (i.e. the projections with respect to the unused clauses) can be deleted and replaced by weakening, thus resulting in a smaller proof.

- If the refutation contains factoring inferences, ψ' will contain constractions operating on ancestors of cut-formulas (note that, in the construction of ψ', resolution inferences become atomic cuts, and factoring inferences become contractions). Since the presence of contractions operating on ancestors of cut-formulas is a major reason for the increase of size and length during cut-elimination, adding such contractions (via factoring) can lead to compression.

However, the original (standard) CERes method [3] also introduces other kinds of redundancies, in the form of unnecessary duplications that occur during the construction of (standard) clause sets and (S-)projections. Therefore, it would be hopeless to expect proof compression by CIRes if it used the standard CERes method. Fortunately, these redundancies can be avoided by using the improved concepts of *swapped clause sets* and *O-projections* that are presented in this paper.

The following subsections define and illustrate all steps of the CIRes method.

2.1 Step 1: Introduction of Atomic Cuts on Top

The first step is the introduction of atomic cuts on the top of the cut-free proof, and it can easily be done according to Definition 2.1.

Definition 2.1 (Introduction of Atomic Cuts)**.** Let φ be a cut-free proof. Then φ^a denotes the proof obtained from φ by replacing every axiom inference with conclusion sequent of the form $A \vdash A$ by a subproof of the form:

$$\dfrac{A \vdash A \qquad A \vdash A}{A \vdash A}\, cut$$

Example 2.1. *Let φ be the cut-free proof below, whose end-sequent was adapted from an instance of a sequence of clause sets used in [7] to show that the resolution calculus can produce significantly shorter proofs than the analytic tableaux calculus.*

$$
\AxiomC{$P^1 \vdash P^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^1, \neg P^1 \vdash$}
\AxiomC{$P^1 \vdash P^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^1, \neg P^1 \vdash$}
\AxiomC{$P^2_- \vdash P^2_-$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^2_-, \neg P^2_- \vdash$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P^2_-, P^1, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P^1, P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{c_l}
\UnaryInfC{$P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\AxiomC{$P^1 \vdash P^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^1, \neg P^1 \vdash$}
\AxiomC{$P^1 \vdash P^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^1, \neg P^1 \vdash$}
\AxiomC{$P^2_- \vdash P^2_-$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^2_-, \neg P^2_- \vdash$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P^2_-, P^1, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P^1, P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{c_l}
\UnaryInfC{$P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\AxiomC{$P^2_+ \vdash P^2_+$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^2_+, \neg P^2_+ \vdash$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P^2_+, P^1 \vee \neg P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{$\vee_l$}
\BinaryInfC{$P^1 \vee P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_-, P^1 \vee \neg P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{c_l}
\UnaryInfC{$P^1 \vee P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_-, P^1 \vee \neg P^2_+, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{c_l}
\UnaryInfC{$P^1 \vee P^2_+, \neg P^1 \vee P^2_-, P^1 \vee \neg P^2_+, \neg P^1 \vee \neg P^2_- \vdash$}
\DisplayProof
$$

Following the first step of the CIRes *method, φ^a is obtained by adding atomic cuts to the leaves of φ:*

$$
\AxiomC{φ^a_l}
\AxiomC{φ^a_r}
\RightLabel{$\vee^1_l$}
\BinaryInfC{$P^1 \vee P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_-, P^1 \vee \neg P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{c_l}
\UnaryInfC{$P^1 \vee P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_-, P^1 \vee \neg P^2_+, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{c_l}
\UnaryInfC{$P^1 \vee P^2_+, \neg P^1 \vee P^2_-, P^1 \vee \neg P^2_+, \neg P^1 \vee \neg P^2_- \vdash$}
\DisplayProof
$$

where φ^a_l is:

$$
\AxiomC{$P^1 \vdash P^1$}
\AxiomC{$P^1 \vdash P^1$}
\dashedLine
\RightLabel{cut}
\BinaryInfC{$P^1 \vdash P^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^1, \neg P^1 \vdash$}
\AxiomC{$P^1 \vdash P^1$}
\AxiomC{$P^1 \vdash P^1$}
\dashedLine
\RightLabel{cut}
\BinaryInfC{$P^1 \vdash P^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^1, \neg P^1 \vdash$}
\AxiomC{$P^2_- \vdash P^2_-$}
\AxiomC{$P^2_- \vdash P^2_-$}
\dashedLine
\RightLabel{cut}
\BinaryInfC{$P^2_- \vdash P^2_-$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^2_-, \neg P^2_- \vdash$}
\RightLabel{$\vee^3_l$}
\BinaryInfC{$P^2_-, P^1, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{$\vee^2_l$}
\BinaryInfC{$P^1, P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{c_l}
\UnaryInfC{$P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\DisplayProof
$$

and φ^a_r is:

$$
\AxiomC{$P^1 \vdash P^1$}
\AxiomC{$P^1 \vdash P^1$}
\dashedLine
\RightLabel{cut}
\BinaryInfC{$P^1 \vdash P^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^1, \neg P^1 \vdash$}
\AxiomC{$P^1 \vdash P^1$}
\AxiomC{$P^1 \vdash P^1$}
\dashedLine
\RightLabel{cut}
\BinaryInfC{$P^1 \vdash P^1$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^1, \neg P^1 \vdash$}
\AxiomC{$P^2_- \vdash P^2_-$}
\AxiomC{$P^2_- \vdash P^2_-$}
\dashedLine
\RightLabel{cut}
\BinaryInfC{$P^2_- \vdash P^2_-$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^2_-, \neg P^2_- \vdash$}
\RightLabel{$\vee^6_l$}
\BinaryInfC{$P^2_-, P^1, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{$\vee^5_l$}
\BinaryInfC{$P^1, P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\RightLabel{c_l}
\UnaryInfC{$P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\AxiomC{$P^2_+ \vdash P^2_+$}
\AxiomC{$P^2_+ \vdash P^2_+$}
\dashedLine
\RightLabel{cut}
\BinaryInfC{$P^2_+ \vdash P^2_+$}
\RightLabel{$\neg_l$}
\UnaryInfC{$P^2_+, \neg P^2_+ \vdash$}
\RightLabel{$\vee^4_l$}
\BinaryInfC{$P^2_+, P^1 \vee \neg P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash$}
\DisplayProof
$$

In φ^a above, each axiom sequent was highlighted with a different color. Other atomic formulas were highlighted with the same color of the axiom from which they descend. Atomic formulas that descend from more than one axiom, in case of contractions, were kept in black color. Each of the six $\vee_l$ inferences was marked with a distinct label from 1 *to* 6 *so that they can be conveniently referred.*

2.2 Step 2: Extraction of the Sruct

The second step is the extraction of the *struct* $\mathcal{S}_{\varphi^a}$ of φ^a. The struct of φ^a contains information about all the relevant atomic sub-formulas (and their polarities) of all cut-formulas of φ^a as well as information about the branching structure of φ^a. A branching inference in φ^a corresponds to either a $\oplus$ or a $\otimes$ connective in $\mathcal{S}_{\varphi^a}$, depending on whether it operates on ancestors of cut-formulas or not. The struct is a compact representation of all information pertinent to cuts in a proof.

Definition 2.2 (Struct)**.** The *struct* $\mathcal{S}_\psi$ of a proof ψ is defined inductively, as follows:

- If ψ consists of an axiom inference ρ with axiom sequent $A \vdash A$ only, then:
  - If only the formula in the succedent is a cut-ancestor, then $\mathcal{S}_\psi \doteq A$.
  - If only the formula in the antecedent is a cut-ancestor, then $\mathcal{S}_\psi \doteq \neg A$.
  - If both formulas of the axiom sequent are cut-ancestors, then $\mathcal{S}_\psi \doteq \neg A \otimes A$.
  - If none of the formulas are cut ancestors, then $\mathcal{S}_\psi \doteq \epsilon_\otimes$.
- If ψ ends with a unary inference ρ, then $\mathcal{S}_\psi \doteq \mathcal{S}_{\psi'}$, where ψ' is the immediate subproof of ψ (i.e. the subproof whose end-sequent is the premise sequent of ρ).
- If ψ ends with a binary inference ρ that operates on cut-ancestors: Let ψ_1 and ψ_2 be the immediate subproofs of ψ. Then:

$$\mathcal{S}_\psi \doteq \mathcal{S}_{\psi_1} \oplus_\rho \mathcal{S}_{\psi_2}$$

- If ψ ends with a binary inference ρ that does not operate on cut-ancestors: Let ψ_1 and ψ_2 be the immediate subproofs of ψ. Then:

$$\mathcal{S}_\psi \doteq \mathcal{S}_{\psi_1} \otimes_\rho \mathcal{S}_{\psi_2}$$

The subscript of a connective may be ommitted, if it is clear or unimportant to which inference it corresponds.

Example 2.2. *The struct $\mathcal{S}_{\varphi^a}$ of φ^a is:*

$$\mathcal{S}_{\varphi^a} \equiv ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_2 ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_3 (P^2_- \oplus \neg P^2_-))) \overset{***}{\otimes}_1 (((P^1 \oplus \neg P^1) \overset{**}{\otimes}_5 ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_6 (P^2_- \oplus \neg P^2_-))) \overset{***}{\otimes}_4 (P^2_+ \oplus \neg P^2_+))$$

It is easy to verify that the connective $\otimes_i$ indeed corresponds to the $\vee_l$ inference marked with label i. The $\oplus$ connectives correspond to the atomic cuts on the top of φ^a and their subscripts have been ommitted. Each $\otimes_i$ connective has been additionally marked with $*$ labels, whose colors are all the colors of ancestors of formulas on which $\vee_l^i$ operates. Although not strictly necessary, these labels are convenient, as shown in Example 2.3.

2.3 Step 3: Construction of the Swapped Clause Set

The connectives $\otimes$ and $\oplus$ can be interpreted as disjunction and conjunction, respectively. In this case, it is possible to prove that the struct is always unsatisfiable [18, 3]. Informally and Intuitively, this fact is also not so hard to see, since the struct contains (the atomic components of) all cut-formulas, which always occur in pairs of opposite polarity. Consequently, the struct contains pairs of dual substructs that cannot be simultaneously satisfied.

In order to refute the unsatisfiable struct by resolution, first it has to be transformed into clause normal form. This could be done by a standard conjunctive normal form transformation, as shown in Definition 2.3. This is essentially what is done in the standard CERes method.

Definition 2.3 ($\underset{s}{\leadsto}$)**.** The standard struct normalization is defined by the following struct rewriting rules:

$$S \otimes (S_1 \oplus \ldots \oplus S_n) \underset{s}{\leadsto} (S \otimes S_1) \oplus \ldots \oplus (S \otimes S_n)$$

$$(S_1 \oplus \ldots \oplus S_n) \otimes S \underset{s}{\leadsto} (S_1 \otimes S) \oplus \ldots \oplus (S_n \otimes S)$$

However, $\underset{s}{\leadsto}$ causes several duplications as $\otimes$ is distributed over $\oplus$, which can make the normalized struct exponentially bigger [1]. These duplications must be avoided, if proof compression is intended. One possible solution to reduce the duplications would be a pre-processing of φ^a that swaps[4] independent inferences that correspond to $\otimes$ upward. The resulting pre-processed proof would have a struct where $\otimes$ connectives already appear more deeply nested and do not need to be distributed over so many $\oplus$ connectives. This pre-processing of proofs would be computationally expensive, though. Fortunately, there is a better alternative, which involves normalizing the struct while implicitly taking into account the possibility of swapping inferences in the corresponding proof, as shown in the struct rewriting system of Definition 2.4.

Definition 2.4 ($\underset{w}{\leadsto}$)**.** Let $\Omega_\rho(\varphi)$ be the set of atomic formula occurrences of φ that are descendants of axioms that contain ancestors of active formulas of ρ. The rewriting rules below distribute $\otimes$ only to those $\oplus$-juncts that contain an occurrence in $\Omega_\rho(\varphi)$. More precisely,, $S_{n+1}, \ldots, S_{n+m}$ and S must contain at least one occurrence from $\Omega_\rho(\varphi)$ each (i.e. there is an atomic substruct S'_{n+k} of S_{n+k} such that $S'_{n+k} \in \Omega_\rho(\varphi)$), and $S_1, \ldots, S_n$ and S_l and S_r should not contain any occurrence from $\Omega_\rho(\varphi)$. Moreover, an innermost rewriting strategy is enforced: only minimal reducible substructs (i.e. structs having no reducible proper substruct) can be rewritten[5].

$$S \otimes_\rho (S_1 \oplus \ldots \oplus S_n \oplus S_{n+1} \oplus \ldots \oplus S_{n+m}) \underset{w}{\leadsto} S_1 \oplus \ldots \oplus S_n \oplus (S \otimes_\rho S_{n+1}) \oplus \ldots \oplus (S \otimes_\rho S_{n+m})$$

$$(S_1 \oplus \ldots \oplus S_n \oplus S_{n+1} \oplus \ldots \oplus S_{n+m}) \otimes_\rho S \underset{w}{\leadsto} S_1 \oplus \ldots \oplus S_n \oplus (S_{n+1} \otimes_\rho S) \oplus \ldots \oplus (S_{n+m} \otimes_\rho S)$$

$$S_l \otimes_\rho S_r \underset{w}{\leadsto} S_l \qquad S_l \otimes_\rho S_r \underset{w}{\leadsto} S_r \qquad S_l \oplus_\rho S_r \underset{w}{\leadsto} S_l \qquad S_l \oplus_\rho S_r \underset{w}{\leadsto} S_r$$

$$S \oplus_\rho S_r \underset{w}{\leadsto} S_r \qquad S_l \oplus_\rho S \underset{w}{\leadsto} S_l$$

The fact that normalization of the struct according to $\underset{w}{\leadsto}$ corresponds to inference swapping taking into account the independence of inferences is stated in Lemma 2.1. This complements the rather technical Definition 2.4 with a more intuitive understanding of the reason why it works.

[4]The proof rewriting system for swapping inferences is shown in Appendix C and defines the relation $\gg$.

[5]Note that m can be equal to zero, in which case the first two rewriting rules simply degenerate to:

$$S \otimes_\rho (S_1 \oplus \ldots \oplus S_n) \underset{w}{\leadsto} S_1 \oplus \ldots \oplus S_n \qquad (S_1 \oplus \ldots \oplus S_n) \otimes_\rho S \underset{w}{\leadsto} S_1 \oplus \ldots \oplus S_n$$

Lemma 2.1 (Correspondence between $\underset{w}{\rightsquigarrow}$ and $\gg$)**.** If φ is skolemized and $\mathcal{S}_\varphi \underset{w}{\rightsquigarrow} S$, then there exists a proof ψ such that $\varphi \gg^* \psi$ and $\mathcal{S}_\psi = S$.

Proof. The proof and example of the correspondence are available in [18]. □

Example 2.3. *$\mathcal{S}_{\varphi^a}$ can be normalized as follows:*

By inspecting φ^a, note that $\vee_l^3$ operates on formulas highlighted in grey and orange. Hence, $\Omega_{\vee_l^3}(\varphi^a)$ contains all formulas highlighted in grey and orange in φ^a (and also some of the formulas in black). Consequently, $\otimes_3$ should only be distributed to substructs that contain formulas highlighted in these colors, and that is why, for convenience, the color label $**$ was added on top of $\otimes_3$. The first rewriting step is shown below:

$$
\begin{array}{rcl}
\mathcal{S}_{\varphi^a} & \equiv & ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_2 ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_3 (P^2_- \oplus \neg P^2_-))) \overset{***}{\otimes}_1 (((P^1 \oplus \neg P^1) \overset{**}{\otimes}_5 ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_6 (P^2_- \oplus \neg P^2_-))) \overset{***}{\otimes}_4 (P^2_+ \oplus \neg P^2_+)) \\
& \underset{w}{\rightsquigarrow} & ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_2 (P^2_- \oplus ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_3 \neg P^2_-))) \overset{***}{\otimes}_1 (((P^1 \oplus \neg P^1) \overset{**}{\otimes}_5 ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_6 (P^2_- \oplus \neg P^2_-))) \overset{***}{\otimes}_4 (P^2_+ \oplus \neg P^2_+))
\end{array}
$$

Note that $(P^1 \oplus \neg P^1)$ (which contains something highlighted in grey) was distributed only to $\neg P^2_-$ (which is highlighted in orange) but not to P^2_- (which is highlighted in neither of those colors but rather in red − orange). Analogously, in the next rewriting step, $\neg P^2_-$ is distributed only to $\neg P^1$, but not to P^1:

$$
\begin{array}{rl}
\ldots \underset{w}{\rightsquigarrow} & ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_2 (P^2_- \oplus ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_3 \neg P^2_-))) \overset{***}{\otimes}_1 (((P^1 \oplus \neg P^1) \overset{**}{\otimes}_5 ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_6 (P^2_- \oplus \neg P^2_-))) \overset{***}{\otimes}_4 (P^2_+ \oplus \neg P^2_+)) \\
\underset{w}{\rightsquigarrow} & ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_2 (P^1 \oplus P^2_- \oplus (\neg P^1 \overset{**}{\otimes}_3 \neg P^2_-))) \overset{***}{\otimes}_1 (((P^1 \oplus \neg P^1) \overset{**}{\otimes}_5 ((P^1 \oplus \neg P^1) \overset{**}{\otimes}_6 (P^2_- \oplus \neg P^2_-))) \overset{***}{\otimes}_4 (P^2_+ \oplus \neg P^2_+))
\end{array}
$$

The rest of the normalization procedure is analogous, as shown below:

$$
\begin{array}{rl}
\ldots \underset{w}{\rightsquigarrow}^* & (P^1 \oplus P^1 \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-)) \overset{***}{\otimes} (((P^1 \oplus \neg P^1) \overset{**}{\otimes} ((P^1 \oplus \neg P^1) \overset{**}{\otimes} (P^2_- \oplus \neg P^2_-))) \overset{***}{\otimes} (P^2_+ \oplus \neg P^2_+)) \\
\underset{w}{\rightsquigarrow}^* & (P^1 \oplus P^1 \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-)) \overset{***}{\otimes} (((P^1 \oplus \neg P^1) \overset{**}{\otimes} (P^1 \oplus P^2_- \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-))) \overset{***}{\otimes} (P^2_+ \oplus \neg P^2_+)) \\
\underset{w}{\rightsquigarrow}^* & (P^1 \oplus P^1 \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-)) \overset{***}{\otimes} ((P^1 \oplus P^1 \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-)) \overset{***}{\otimes} (P^2_+ \oplus \neg P^2_+)) \\
\underset{w}{\rightsquigarrow}^* & (P^1 \oplus P^1 \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-)) \overset{***}{\otimes} ((P^1 \overset{***}{\otimes} \neg P^2_+) \oplus (P^1 \overset{***}{\otimes} \neg P^2_+) \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-) \oplus P^2_+) \\
\underset{w}{\rightsquigarrow}^* & ((P^1 \overset{***}{\otimes} P^2_+) \oplus (P^1 \overset{***}{\otimes} P^2_+) \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-)) \oplus ((P^1 \overset{***}{\otimes} \neg P^2_+) \oplus (P^1 \overset{***}{\otimes} \neg P^2_+) \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-)) \\
\equiv & (P^1 \overset{***}{\otimes} P^2_+) \oplus (P^1 \overset{***}{\otimes} P^2_+) \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-) \oplus (P^1 \overset{***}{\otimes} \neg P^2_+) \oplus (P^1 \overset{***}{\otimes} \neg P^2_+) \oplus (\neg P^1 \overset{**}{\otimes} P^2_-) \oplus (\neg P^1 \overset{**}{\otimes} \neg P^2_-)
\end{array}
$$

Definition 2.5 (Swapped Clause Set)**.** A *swapped clause set*[6] $C^W_{\varphi|S}$ of a proof φ with respect to a $\underset{w}{\rightsquigarrow}$-normal-form S of $\mathcal{S}_\varphi$ is the set of clauses (in sequent notation) obtained from S by interpreting $\otimes$ as $\vee$ and $\oplus$ as $\wedge$. In cases where a proof φ has only one cut-pertinent swapped clause set, it can be denoted simply as C^W_φ.

Example 2.4. *The swapped clause set $C^W_{\varphi^a}$ is shown below. Note how each $\oplus$-junct of the normal form of $\mathcal{S}_{\varphi^a}$ shown in Example 2.3 corresponds to a clause in $C^W_{\varphi^a}$.*

$$
C^W_{\varphi^a} \equiv \{\ \vdash P^1, P^2_+ \ ;\ \vdash P^1, P^2_+ \ ;\ P^1 \vdash P^2_- \ ;\ P^1, P^2_- \vdash \ ;\ P^2_+ \vdash P^1 \ ;\ P^2_+ \vdash P^1 \ ;\ P^1 \vdash P^2_- \ ;\ P^1, P^2_- \vdash \ \}
$$

2.4 Step 4: Refutation of the Swapped Clause Set by Resolution

The fourth step is the search for a resolution refutation of the swapped clause set.

Example 2.5. *$C^W_{\varphi^a}$ can be refuted by the refutation δ below:*

[6] Historically, swapped clause sets were developed during an attempt to give a more intuitive definition for *profile clause sets* [11]. The deeper understanding acquired during this attempt allowed the development of swapped clause sets, which are technically better than profile clause sets in handling proofs with weakening inferences [18].

$$\dfrac{\dfrac{\dfrac{\vdash P^1, P^2_+ \qquad P^2_+ \vdash P^1}{\vdash P^1, P^1}\, r}{\vdash P^1}\, f_r \qquad \dfrac{\dfrac{P^1 \vdash P^2_- \qquad P^1, P^2_- \vdash}{P^1, P^1 \vdash}\, r}{P^1 \vdash}\, f_r}{\vdash}\, r$$

2.5 Step 5: Construction of O-Projections

A projection's purpose is to replace a leaf in a refutation of the clause set. Therefore, its end-sequent must contain the leaf's clause as a subsequent. Moreover, if we consider the composition of projections on the refutation described in Subsection 2.6, it is clear that any other formula F in the end-sequent of a projection will also appear in the end-sequent of the proof with atomic cuts produced by CIRes. Since the end-sequent of the proof produced by CIRes should be the same as the end-sequent of the original proof without cuts, it must not be the case that the end-sequent of a projection contains a formula F that does not already appear in the end-sequent of the original proof without cuts. These conditions are formally expressed in Definition 2.6.

Definition 2.6 (Projection)**.** Let φ be a proof with end-sequent $\Gamma \vdash \Delta$ and $c \equiv \Gamma_c \vdash \Delta_c \in C_\varphi$. Any cut-free proof of $(\Gamma', \Gamma_c \vdash \Delta', \Delta_c)\sigma$, where $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$ and σ is a substitution, is a *projection* of φ with respect to c.

The standard method for the construction of projections is usually presented together with descriptions of the CERes method [2, 5, 3, 12]. It can be easily described, but unfortunately results in redundant projections, because parts of φ tend to appear several times in different projections of φ, even though it would suffice if they appeared in only one of these projections. Projections of this standard but redundant kind are here called *S-projections*. This paper describes an alternative method that constructs so-called *O-projections* (Definition 2.9). They are less redundant and thus essential for compressing proofs via CIRes. Their construction relies on the auxiliary Y rule (Definition 2.7) and on the concept of *axiom-linkage* (Definition 2.8).

Definition 2.7 (Y Rule)**.** The Y rule of inference is shown below:

$$\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1 \vdash \Delta_1\end{array} \qquad \ldots \qquad \begin{array}{c}\varphi_n\\ \Gamma_n \vdash \Delta_n\end{array}}{\Gamma_1, \ldots, \Gamma_n \vdash \Delta_1 \ldots, \Delta_n}\, Y$$

Definition 2.8 (Axiom Linkage)**.** Two atomic (sub)formulas A_1 and A_2 in a proof φ are *axiom-linked*[7] if and only if they have ancestors in the same axiom sequent.

Definition 2.9 (O-Projection)**.** The *O-projection*[8] $\lfloor \varphi \rfloor^O_c$ of the proof φ with respect to the clause c is constructed according to the following steps:

1. replace inferences that operate on cut-ancestors by Y-inferences.

2. replace by Y-inferences also those inferences such that none of its active formulas is axiom-linked to a formula of φ appearing in c.

3. delete formulas that are not axiom-linked to the formulas appearing in c.

[7] By definition of axiom-linkage, it is clear that formulas highlighted with the same color in Example 2.1 are axiom-linked to each other.

[8] A technically more detailed definition of O-projection is available in [18].

4. if the previous step deleted an auxiliary formula of an inference, fix the inference by adding a weakening inference that re-introduces the auxiliary formula.

5. eliminate the Y-inferences, according to Definition 2.10.

Definition 2.10 (Y-Elimination)**.** The elimination of Y inferences follows the proof rewriting rule shown below. For the rewriting rule to be applicable, φ_j is required to be a correct proof containing no Y-inferences.

$$\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1 \vdash \Delta_1\end{array} \quad \dots \quad \begin{array}{c}\varphi_n\\ \Gamma_n \vdash \Delta_n\end{array}}{\Gamma_1,\dots,\Gamma_n \vdash \Delta_1 \dots,\Delta_n}\ Y$$

$$\Downarrow$$

$$\begin{array}{c}\varphi_j\\ \Gamma_j \vdash \Delta_j\end{array} \atop \overline{\overline{\Gamma_1,\dots,\Gamma_n \vdash \Delta_1 \dots,\Delta_n}}\ w^*$$

Theorem 2.1 (Correctness of O-Projections)**.** If φ is a skolemized proof, then $\lfloor\varphi\rfloor_c^O$ is a projection of φ with respect to c.

Proof. In order to prove this theorem, it is necessary to show that the algorithm for construction of O-projections described in Definition 2.9 outputs a proof that satisfies the requirements expressed in Definition 2.6. A detailed technical proof is available in [18], and only a few informal remarks are presented here. Note that $\lfloor\varphi\rfloor_c^O$ is obviously cut-free, because of step 1 in Definition 2.9. Step 1 also guarantees that c appears as a subsequent of the end-sequent of $\lfloor\varphi\rfloor_c^O$: the atoms of c originate from atomic formulas that occur in axiom sequents and are ancestors of cut-formulas, and since all inferences that operate on ancestors of cut-formulas are replaced by Y-inferences, these atoms are free to propagate down to the end-sequent of $\lfloor\varphi\rfloor_c^O$ (i.e. they will not be used by propositional or quantifier inferences to compose more complex formulas and they will not be consumed by cut inferences, because all these inferences are replaced). Step 3 guarantees that propagated atoms of other clauses of C_φ are deleted from the end-sequent of $\lfloor\varphi\rfloor_c^O$, so that only the propagated atoms of c remain. Moreover, note that $\lfloor\varphi\rfloor_c^O$ will still contain those inferences of φ that operate on formulas that are axiom-linked to formulas of c and are not ancestors of cut-formulas. If these formulas are not ancestors of cut-formulas, they have (also axiom-linked) descendants occurring in the end-sequent of φ, which were not deleted by any step in the construction of the projection. Therefore, the end-sequent of $\lfloor\varphi\rfloor_c^O$ is of the form $\Gamma',\Gamma_c \vdash \Delta',\Delta_c$, where $\Gamma' \vdash \Delta'$ is the subsequent of the end-sequent of φ whose formulas are axiom-linked to formulas that appear in c, and $\Gamma_c \vdash \Delta_c = c$, since other propagated atoms are deleted by step 3. φ is required to be skolemized in order to prevent violations of eigen-variable conditions by the atoms that are propagated down after step 1. □

Example 2.6. *In the construction of the projection $\lfloor\varphi^a\rfloor^O_{\vdash P^1,P^2_+}$, the first four steps result in the (partial) proof with Y inferences shown below. Note that the only remaining formulas are those axiom linked to P^1 and P^2_+. All other were deleted. And the only remaining non-Y inference is $\vee^1_l$, because it is the only one that operates on formulas that are axiom-linked to P^1 and P^2_+.*

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P^1 \vdash P^1 \quad \vdash}{P^1 \vdash P^1}\,Y}{P^1 \vdash P^1}\,Y \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \quad \vdash}{\vdash}\,Y}{\vdash}\,Y \quad \dfrac{\dfrac{\vdash \quad \vdash}{\vdash}\,Y}{\vdash}\,Y}{\vdash}\,Y}{}}{P^1 \vdash P^1}\,Y}{P^1 \vdash P^1}\,Y \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \quad \vdash}{\vdash}\,Y}{\vdash}\,Y \quad \dfrac{\dfrac{\dfrac{\vdash \quad \vdash}{\vdash}\,Y}{\vdash}\,Y \quad \dfrac{\dfrac{\vdash \quad \vdash}{\vdash}\,Y}{\vdash}\,Y}{\vdash}\,Y}{\vdash}\,Y}{\vdash}\,Y \quad \dfrac{\dfrac{P^2_+ \vdash P^2_+ \quad \vdash}{P^2_+ \vdash P^2_+}\,Y}{P^2_+ \vdash P^2_+}\,Y}{P^2_+ \vdash P^2_+}\,Y}{P^1 \vee P^2_+ \vdash P^1, P^2_+}\,\vee_l^1}{P^1 \vee P^2_+ \vdash P^1, P^2_+}\,Y}{P^1 \vee P^2_+ \vdash P^1, P^2_+}\,Y
$$

The last step in the construction of O-projections is the elimination of Y-inferences, and it gives the O-projection $\lfloor\varphi^a\rfloor^O_{\vdash P^1, P^2_+}$ shown below. The projections with respect to the other three clauses that appear in the leaves of the refutation are also shown below:

$\lfloor\varphi^a\rfloor^O_{\vdash P^1, P^2_+}$:

$$
\dfrac{P^1 \vdash P^1 \qquad P^2_+ \vdash P^2_+}{P^1 \vee P^2_+ \vdash P^1, P^2_+}\,\vee_l
$$

$\lfloor\varphi^a\rfloor^O_{P^2_+ \vdash P^1}$:

$$
\dfrac{P^1 \vdash P^1 \qquad \dfrac{P^2_+ \vdash P^2_+}{P^2_+, \neg P^2_+ \vdash}\,\neg_l}{P^2_+, P^1 \vee \neg P^2_+ \vdash P^1}\,\vee_l
$$

$\lfloor\varphi^a\rfloor^O_{P^1 \vdash P^2_-}$:

$$
\dfrac{\dfrac{P^1 \vdash P^1}{P^1, \neg P^1 \vdash}\,\neg_l \qquad P^2_- \vdash P^2_-}{P^1, \neg P^1 \vee P^2_- \vdash P^2_-}\,\vee_l
$$

$\lfloor\varphi^a\rfloor^O_{P^1, P^2_- \vdash}$:

$$
\dfrac{\dfrac{P^1 \vdash P^1}{P^1, \neg P^1 \vdash}\,\neg_l \qquad \dfrac{P^2_- \vdash P^2_-}{P^2_-, \neg P^2_- \vdash}\,\neg_l}{P^1, P^2_-, \neg P^1 \vee \neg P^2_- \vdash}\,\vee_l
$$

2.6 Step 6: Composing the O-Projections on top of the Refutation

The last step is the replacement of each leaf of the ground[9] refutation by its corresponding projection, and the renaming of resolution inferences to cuts and of factoring inferences to contractions.

Example 2.7. *The final proof with atomic cuts, obtained by composing the refutation and the projections, is* $\mathrm{CIRes}^O_W(\varphi, \delta)$ *shown below:*

$$
\dfrac{\dfrac{\dfrac{\dfrac{P^1 \vdash P^1 \quad P^2_+ \vdash P^2_+}{P^1 \vee P^2_+ \vdash P^1, P^2_+}\,\vee_l \quad \dfrac{P^1 \vdash P^1 \quad \dfrac{P^2_+ \vdash P^2_+}{P^2_+, \neg P^2_+ \vdash}\,\neg_l}{P^2_+, P^1 \vee \neg P^2_+ \vdash P^1}\,\vee_l}{P^1 \vee P^2_+, P^1 \vee \neg P^2_+ \vdash P^1, P^1}\,cut}{P^1 \vee P^2_+, P^1 \vee \neg P^2_+ \vdash P^1}\,c_r \quad \dfrac{\dfrac{\dfrac{\dfrac{P^1 \vdash P^1}{P^1, \neg P^1 \vdash}\,\neg_l \quad P^2_- \vdash P^2_-}{P^1, \neg P^1 \vee P^2_- \vdash P^2_-}\,\vee_l \quad \dfrac{\dfrac{P^1 \vdash P^1}{P^1, \neg P^1 \vdash}\,\neg_l \quad \dfrac{P^2_- \vdash P^2_-}{P^2_-, \neg P^2_- \vdash}\,\neg_l}{P^1, P^2_-, \neg P^1 \vee \neg P^2_- \vdash}\,\vee_l}{P^1, P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash}\,cut}{P^1, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash}\,c_r}{P^1 \vee P^2_+, P^1 \vee \neg P^2_+, \neg P^1 \vee P^2_-, \neg P^1 \vee \neg P^2_- \vdash}\,cut
$$

[9] Although the example shown here is purely propositional, CIRes works in predicate logic too. In this case, it is necessary to ground the resolution refutation by applying a substitution that is the composition of all most general unifiers used in the refutation. Only then the resolution inferences can be converted to atomic cuts. The grounding substitution must be applied to the projections too.

Table 1 compares the sizes of φ and $\mathrm{CIRes}^{O}_{W}(\varphi,\delta)$, and thus shows that CIRes *is indeed able to compress proofs. Three different measures are used:* proof length*, which is the number of inferences in the proof;* symbolic proof size*, which counts the total number of symbols in formulas occurring in the proof; and* atomic proof size*, which counts only the total number of predicate symbols in formulas occurring in the proof.*

Table 1: Compression by CIRes

	φ	$\mathrm{CIRes}^{O}_{W}(\varphi,\delta)$
Proof Length (number of inferences)	17	13
Symbolic Proof Size (number of formula symbols)	169	105
Atomic Proof Size (number of atoms)	97	70

3 Exponential Proof Compression

The following lemmas and theorems use a set of disjunctions D_m that is associated with the complete binary tree of depth m, as described in [7, 17]. D_m contains 2^m disjunctions of the form[10] $\circ P^1 \vee \circ P^2_{\pm} \vee \circ P^3_{\pm\pm} \vee \ldots \vee \circ P^m_{\pm\ldots\pm}$, where $\circ$ is either empty or $\neg$ and the i-th $\pm$ is either $+$, if the $\circ$ preceding $P^i_{\pm\ldots\pm}$ is empty, or $-$, if the $\circ$ preceding $P^i_{\pm\ldots\pm}$ is $\neg$. For example, $D_2 = \{P^1 \vee P^2_+, \neg P^1 \vee P^2_-, P^1 \vee \neg P^2_+, \neg P^1 \vee \neg P^2_-\}$. C_m is defined as the set of clauses corresponding to the disjunctions of D_m (e.g. $D_2 = \{\vdash P^1, P^2_+ \ ; \ P^1 \vdash P^2_- \ ; \ P^2_+ \vdash P^1 \ ; \ P^1, P^2_- \vdash\}$). And T_m is defined as the sequent having all the disjunctions of C_m in its antecedent (e.g. $T_2 = P^1 \vee P^2_+, \neg P^1 \vee P^2_-, P^1 \vee \neg P^2_+, \neg P^1 \vee \neg P^2_- \vdash$).

P^1 $\qquad$ $\neg P^1$

P^2_+ $\qquad$ $\neg P^2_+$ $\qquad$ P^2_- $\qquad$ $\neg P^2_-$

Note that T_2 is exactly the end-sequent of the proofs considered in the examples of the previous section. The assymptotic results about the compression achievable by CIRes are obtained by quantitatively analyzing what happens in the general case, when CIRes is applied to T_m. The general phenomenon is essentially the same as what has been observed for T_2, and hence the examples of the previous section are helpful for the comprehension of the lemmas in this section.

Lemma 3.1. Let ψ_m be a shortest analytic tableaux refutation of D_m. Then $length(\psi_m) > 2^{k_1 2^m}$, for some positive rational constant k_1.

Proof. This lemma was mentioned in [7] and then proved in [17]. □

Lemma 3.2. Let φ_m be the shortest cut-free sequent calculus proof of T_m corresponding to ψ_m. Then $length(\varphi_m) > 2^{k_2 2^m}$,for some positive rational constant k_2.

[10] Parentheses have been ommited from these disjunctions, and the $\vee$ connective is assumed to be left-associative.

Proof. This lemma follows immediately from Lemma 3.1 and from the fact that cut-free sequent calculus p-simulates analytic tableaux [17]. □

Lemma 3.3. Let δ_m be the shortest resolution refutation of C_m. Then $length(\delta_m) < 2^{k_3 m}$, for some positive rational constant k_3.

Proof. This lemma is mentioned without proof in [7]. Its proof is easy, though. δ_m can be constructed by resolving first the literals that correspond to the deepest nodes in the complete binary tree that generates D_m and C_m, then resolving the literals that correspond to the nodes on the level immediately above, and so on, until all literals have been resolved. In this way, it is clear that the number of resolution and factoring inferences in δ_m is linearly related to the number of nodes in the binary tree, which is exponential in m. □

Lemma 3.4. $C_m \subseteq C^W_{\varphi^a_m}$.

Proof. Let $c_d \in C_m$. By definition of C_m, D_m and T_m, there is a disjunction $d \doteq L_1 \vee^1 \ldots \vee^{m-1} L_m$ in the antecedent of the end-sequent T_m of the proof φ^a_m such that c_d is the clause form of d in sequent notation. Let A_i be one of the atomic ancestors of L_i occurring in an axiom sequent s_i of φ^a_m. Let A'_i be the formula occurring in the other cedent of s_i. Note that:

- A_i and A'_i are syntactically equal, by definition of axiom sequents.
- A'_i is a cut-ancestor.
- A'_i occurs in the succedent of s_i, if L_i is a positive literal, and in the antecedent, otherwise.

Let S_i be the substruct of $\mathcal{S}_{\varphi^a_m}$ corresponding to the axiom rule having conclusion sequent s_i. By definition, $S_i = A'_i = L_i$, if A'_i occurs in the succedent of s_i, and $S_i = \neg A'_i = L_i$, otherwise. Let ρ_j be the $\vee_l$ inference in φ^a_m which operates on descendents of $A_1, \ldots, A_m$ and introduces the connective $\vee^j$ in the disjunction d. Let $\otimes_j$ be the connective in $\mathcal{S}_{\varphi^a_m}$ that corresponds to ρ_j. By Definition 2.4, the normalization of $\mathcal{S}_{\varphi^a_m}$ is such that $\otimes_j$ is distributed to substructs containing formulas that are axiom-linked to ancestors of formulas on which ρ_j operates (i.e. containing A'_1, $\ldots, A'_m$). Consequently, the normal form S of $\mathcal{S}_{\varphi^a_m}$ contains the substruct $S_d \doteq S_1 \otimes_1 \ldots \otimes_{m-1} S_m$. When S is transformed to $C^W_{\varphi^a_m}$, S_d becomes the clause c_d, because $S_i = L_i$, for all i, and $\otimes$ is interpreted as $\vee$. Therefore, $c_d \in C^W_{\varphi^a_m}$ and hence $C_m \subseteq C^W_{\varphi^a_m}$. □

Lemma 3.5. δ_m is a refutation of $C^W_{\varphi^a_m}$.

Proof. This lemma follows immediately from Lemma 3.4. □

Lemma 3.6. Let $c \in C^W_{\varphi^a_m}$. Then $length(\lfloor \varphi^a_m \rfloor^O_c) < k_4 m$, for some positive constant k_4.

Proof. By definition, the O-projection $\lfloor \varphi^a_m \rfloor^O_c$ contains only those inferences of φ^a_m that operate on descendants of axiom sequents that contain occurences of c. These are the inferences that construct one of the disjunctions in the end-sequent T_m of φ^a_m, namely the disjunction whose clause form in sequent notation is equal to c. c has exactly m literals, and thus $\lfloor \varphi^a_m \rfloor^O_c$ contains exactly $m-1$ inferences of $\vee_l$ kind. Since literals can appear negated, $\lfloor \varphi^a_m \rfloor^O_c$ can contain at most m inferences of $\neg_l$ kind. No other inferences appear in $\lfloor \varphi^a_m \rfloor^O_c$. Therefore, $length(\lfloor \varphi^a_m \rfloor^O_c) < 2m-1$. □

Theorem 3.1 (Exponential Proof Compression via CIRes)**.** There exists a sequence of sequents T_m such that:

- if φ_m is a sequence of shortest cut-free proofs of T_m, then $length(\varphi_m) > 2^{k_5 2^m}$ (for some positive constant k_5).

- there exists δ_m such that $length(\text{CIRes}(\varphi_m, \delta_m)) < m.2^{k_6 m}$ (for some positive constant k_6).

Proof. The first item of this theorem is just Lemma 3.2. For the second item, let δ_m be the shortest refutation of C_m, as in Lemma 3.3. By lemma 3.5, δ_m is also a refutation of $C^W_{\varphi^a_m}$, and hence it can be used in the construction of the proof with atomic cuts by CIRes. Then note that $\text{CIRes}(\varphi_m, \delta_m)$ is the composition of δ_m, whose length is exponentially upperbounded (Lemma 3.3), and 2^m O-projections of linear size (as in Lemma 3.6). Therefore:

$$length(\text{CIRes}(\varphi_m, \delta_m)) < 2^{k_3 m} + 2^m(2m-1) < m.2^{k_6 m}$$

for some constant k_6. □

4 Conclusions

This paper has introduced the CIRes method of cut-introduction and shown that it can compress proofs exponentially. This was only possible with the development of swapped clause sets and O-projections, which are less redundant than the standard clause sets and projections traditionally used by CERes. These new concepts could be employed for cut-elimination as well.

The further development of CIRes can proceed in various directions. Firstly, similarly to what has already been done for cut-elimination [18], the swapped clause set could be enriched with additional information from the proof, and this information could then be used to define refinements of the resolution calculus in order to facilitate the search for refutations and, consequently, the introduction of cuts.

Secondly, O-projections and the method for combining them with the refutation could still be improved. In the example considered in this paper, the shortest proof with cuts was a proof whose atomic cuts all occur in the bottom, which is particularly suitable for a method like CIRes, that outputs proofs with cuts in the bottom. However, in other cases (i.e. when the optimal proof with atomic cuts is such that the atomic cuts do not occur in the bottom of the proof), CIRes might produce sub-optimally compressed proofs, because the O-projections will contain redundancies that are only necessary because CIRes currently requires the atomic cuts to be in the very bottom and the projections to be on the top. This indicates that CIRes could be improved by developing different notions of projections and more flexible ways of composing them with the refutation. However, this is highly non-trivial.

Finally, much more significant (i.e. non-elementary) compression could in principle be obtained via introduction of quantified cuts. The CIRes method described in this chapter introduces only atomic cuts and is therefore just a first step toward the harder task of introducing complex quantified cuts. An intermediary step could be the introduction of propositional cuts, possibly by using definitional and swapped definitional clause sets [18]. But even then, (sub-optimal) compressive quantified-cut-introduction would still be a distant goal, and an algorithm that would generally guarantee optimal compression is forever out of reach; it cannot exist, due to the undecidability results in [6].

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Appendix A: Sequent Calculus

The inference rules of the sequent calculus used in this paper are shown below. The sequent below the line of an inference rule is its *conclusion*, while the sequents above the line are its *premises*. The formulas highlighted in red color are called *main* formulas of the rules, while the formulas highlighted in blue color are called *auxiliary* formulas. Main and auxiliary formulas are called *active*. An inference is said to *operate* on its active formulas. Note that CERes and the methods described in this paper are robust and work with other kinds of sequent calculi, as long as weakening and contraction are available in some (possibly implicit) form.

- **The Axiom Rule**:

$$\frac{}{A \vdash A}\ axiom$$

  where A is any atomic formula.

- **Propositional Rules**:

$$\frac{F_1, F_2, \Gamma \vdash \Delta}{F_1 \wedge F_2, \Gamma \vdash \Delta}\ \wedge_l \qquad \frac{\Gamma_1 \vdash \Delta_1, F_1 \quad \Gamma_2 \vdash \Delta_2, F_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, F_1 \wedge F_2}\ \wedge_r \qquad \frac{F_1, \Gamma_1 \vdash \Delta_1 \quad F_2, \Gamma_2 \vdash \Delta_2}{F_1 \vee F_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \vee_l \qquad \frac{\Gamma \vdash \Delta, F}{\neg F, \Gamma \vdash \Delta}\ \neg_l$$

$$\frac{\Gamma_1 \vdash \Delta_1, F_1 \quad F_2, \Gamma_2 \vdash \Delta_2}{F_1 \rightarrow F_2, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ \rightarrow_l \qquad \frac{F_1, \Gamma \vdash \Delta, F_2}{\Gamma \vdash \Delta, F_1 \rightarrow F_2}\ \rightarrow_r \qquad \frac{\Gamma \vdash \Delta, F_1, F_2}{\Gamma \vdash \Delta, F_1 \vee F_2}\ \vee_r \qquad \frac{F, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg F}\ \neg_r$$

- **Structural Rules (Weakening and Contraction)**:

$$\frac{\Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\ w_l \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, F}\ w_r \qquad \frac{F, F, \Gamma \vdash \Delta}{F, \Gamma \vdash \Delta}\ c_l \qquad \frac{\Gamma \vdash \Delta, F, F}{\Gamma \vdash \Delta, F}\ c_r$$

- **The Cut Rule**:

$$\frac{\Gamma_1 \vdash \Delta_1, F \quad F, \Gamma_2 \vdash \Gamma_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ cut$$

- **Quantifier Rules**:

$$\frac{F\{x \leftarrow t\}, \Gamma \vdash \Delta}{(\forall x)F, \Gamma \vdash \Delta}\ \forall_l \qquad \frac{\Gamma \vdash \Delta, F\{x \leftarrow \alpha\}}{\Gamma \vdash \Delta, (\forall x)F}\ \forall_r \qquad \frac{F\{x \leftarrow \alpha\}, \Gamma \vdash \Delta}{(\exists x)F, \Gamma \vdash \Delta}\ \exists_l \qquad \frac{\Gamma \vdash \Delta, F\{x \leftarrow t\}}{\Gamma \vdash \Delta, (\exists x)F}\ \exists_r$$

  For the $\forall_r$ and the $\exists_l$ rules, the variable α must not occur in Γ nor in Δ nor in F. This is the *eigenvariable condition*. For the $\forall_l$ and the $\exists_r$ rules the term t must not contain a variable that is bound in F.

Appendix B: Reductive Cut-Elimination Proof Rewriting Rules

Reductive cut-elimination methods were introduced by Gentzen and can be described as a proof rewriting system. The rewriting rules for the sequent calculus used in this paper are shown below.

Definition 4.1 ($\triangleright_a$)**.** Cut-elimination over axiom inferences:

$$\frac{A \vdash A \qquad \begin{array}{c}\varphi_r\\ A, \Pi \vdash \Lambda\end{array}}{A, \Pi \vdash \Lambda}\ cut \qquad\qquad \frac{\begin{array}{c}\varphi_l\\ \Gamma \vdash \Delta, A\end{array} \qquad A \vdash A}{\Gamma, \vdash \Delta, A}\ cut$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{array}{c}\varphi_r\\ A, \Pi \vdash \Lambda\end{array} \qquad\qquad\qquad\qquad \begin{array}{c}\varphi_l\\ \Gamma \vdash \Delta, A\end{array}$$

Definition 4.2 ($\triangleright_{r_1}$)**.** Upward swapping of cuts over unary inferences (unary rank reduction):

$$\dfrac{\overset{\varphi_l}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_r}{A, \Pi' \vdash \Lambda'}}{A, \Pi \vdash \Lambda}\,\rho}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut \qquad\qquad \dfrac{\dfrac{\overset{\varphi_l}{\Gamma' \vdash \Delta', A}}{\Gamma \vdash \Delta, A}\,\rho \qquad \overset{\varphi_r}{A, \Pi \vdash \Lambda}}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_l}{\Gamma \vdash \Delta, A} \qquad \overset{\varphi_r}{A, \Pi' \vdash \Lambda'}}{\Gamma, \Pi' \vdash \Delta, \Lambda'}\,cut}{\Gamma, \Pi \vdash \Delta, \Lambda}\,\rho \qquad\qquad \dfrac{\dfrac{\overset{\varphi_l}{\Gamma' \vdash \Delta', A} \qquad \overset{\varphi_r}{A, \Pi \vdash \Lambda}}{\Gamma', \Pi \vdash \Delta', \Lambda}\,cut}{\Gamma, \Pi \vdash \Delta, \Lambda}\,\rho$$

Definition 4.3 ($\rhd_{r_2}$). Upward swapping of cuts over binary inferences (binary rank reduction):

$$\dfrac{\overset{\varphi_l}{\Pi \vdash \Lambda, A} \qquad \dfrac{\overset{\varphi_1}{A, \Gamma_1 \vdash \Delta_1} \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2}}{A, \Gamma \vdash \Delta}\,\rho}{\Pi, \Gamma \vdash \Lambda, \Delta}\,cut \qquad\qquad \dfrac{\overset{\varphi_l}{\Pi \vdash \Lambda, A} \qquad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1} \qquad \overset{\varphi_2}{A, \Gamma_2 \vdash \Delta_2}}{A, \Gamma \vdash \Delta}\,\rho}{\Pi, \Gamma \vdash \Lambda, \Delta}\,cut$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_l}{\Pi \vdash \Lambda, A} \qquad \overset{\varphi_1}{A, \Gamma_1 \vdash \Delta_1}}{\Pi, \Gamma_1 \vdash \Lambda, \Delta_1}\,cut \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2}}{\Pi, \Gamma \vdash \Lambda, \Delta}\,\rho \qquad\qquad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1} \qquad \dfrac{\overset{\varphi_l}{\Pi \vdash \Lambda, A} \qquad \overset{\varphi_2}{A, \Gamma_2 \vdash \Delta_2}}{\Pi, \Gamma_2 \vdash \Lambda, \Delta_2}\,cut}{\Pi, \Gamma \vdash \Lambda, \Delta}\,\rho$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1, A} \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2}}{\Gamma \vdash \Delta, A}\,\rho \qquad \overset{\varphi_r}{A, \Pi \vdash \Lambda}}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut \qquad\qquad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1} \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2, A}}{\Gamma \vdash \Delta, A}\,\rho \qquad \overset{\varphi_r}{A, \Pi \vdash \Lambda}}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1, A} \qquad \overset{\varphi_r}{A, \Pi \vdash \Lambda}}{\Gamma_1, \Pi \vdash \Delta_1, \Lambda}\,cut \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2}}{\Gamma, \Pi \vdash \Delta, \Lambda}\,\rho \qquad\qquad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1} \qquad \dfrac{\overset{\varphi_2}{\Gamma_2 \vdash \Delta_2, A} \qquad \overset{\varphi_r}{A, \Pi \vdash \Lambda}}{\Gamma_2, \Pi \vdash \Delta_2, \Lambda}\,cut}{\Gamma, \Pi \vdash \Delta, \Lambda}\,\rho$$

Definition 4.4 ($\rhd_{p_\wedge}$). Reduction of complexity of a cut-formula having $\wedge$ as shallowest connective (grade reduction):

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1, B} \qquad \overset{\varphi_2}{\Gamma_2 \vdash \Delta_2, C}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, B \wedge C}\,\wedge_r \qquad \dfrac{\overset{\varphi_r}{B, C, \Pi \vdash \Lambda}}{B \wedge C, \Pi \vdash \Lambda}\,\wedge_l}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda}\,cut$$

$$\Downarrow$$

$$\dfrac{\overset{\varphi_2}{\Gamma_2 \vdash \Delta_2, C} \qquad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1, B} \qquad \overset{\varphi_r}{B, C, \Pi \vdash \Lambda}}{C, \Gamma_1, \Pi \vdash \Delta_1, \Lambda}\,cut}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda}\,cut$$

Definition 4.5 ($\rhd_{p_\vee}$)**.** Reduction of complexity of a cut-formula having $\vee$ as shallowest connective (grade reduction):

$$\dfrac{\dfrac{\overset{\varphi_l}{\Pi \vdash \Lambda, B, C}}{\Pi \vdash \Lambda, B \vee C}\,\vee_r \qquad \dfrac{\overset{\varphi_1}{B, \Gamma_1 \vdash \Delta_1} \qquad \overset{\varphi_2}{C, \Gamma_2 \vdash \Delta_2}}{B \vee C, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,\vee_l}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda}\,cut$$

$$\Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_l}{\Pi \vdash \Lambda, B, C} \qquad \overset{\varphi_2}{C, \Gamma_2 \vdash \Delta_2}}{\Pi, \Gamma_2 \vdash \Delta_2, \Lambda, B}\,cut \qquad \overset{\varphi_1}{B, \Gamma_1 \vdash \Delta_1}}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda}\,cut$$

Definition 4.6 ($\rhd_{p_\rightarrow}$)**.** Reduction of complexity of a cut-formula having $\rightarrow$ as shallowest connective (grade reduction):

$$\dfrac{\dfrac{\overset{\varphi_l}{B, \Pi \vdash \Lambda, C}}{\Pi \vdash \Lambda, B \rightarrow C}\,\rightarrow_r \qquad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1, B} \qquad \overset{\varphi_2}{C, \Gamma_2 \vdash \Delta_2}}{B \rightarrow C, \Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\,\rightarrow_l}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda}\,cut$$

$$\Downarrow$$

$$\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1, B} \qquad \dfrac{\overset{\varphi_l}{B, \Pi \vdash \Lambda, C} \qquad \overset{\varphi_2}{C, \Gamma_2 \vdash \Delta_2}}{B, \Pi, \Gamma_2 \vdash \Delta_2, \Lambda}\,cut}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda}\,cut$$

Definition 4.7 ($\rhd_{p_\neg}$)**.** Reduction of complexity of a cut-formula having $\neg$ as shallowest connective (grade reduction):

$$\dfrac{\dfrac{\overset{\varphi_l}{B, \Gamma \vdash \Delta}}{\Gamma \vdash \Delta, \neg B}\,\neg_r \qquad \dfrac{\overset{\varphi_r}{\Pi \vdash \Lambda, B}}{\neg B, \Pi \vdash \Lambda}\,\neg_l}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut$$

$$\Downarrow$$

$$\dfrac{\overset{\varphi_r}{\Pi \vdash \Lambda, B} \qquad \overset{\varphi_l}{B, \Gamma \vdash \Delta}}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut$$

Definition 4.8 ($\rhd_{q_\forall}$)**.** Reduction of complexity of a cut-formula having a universal quantifier at its shallowest level:

$$\dfrac{\dfrac{\overset{\varphi_l}{\Gamma \vdash \Delta, B\{x \leftarrow \alpha\}}}{\Gamma \vdash \Delta, \forall x B}\,\forall_r \qquad \dfrac{\overset{\varphi_r}{B\{x \leftarrow t\}, \Pi \vdash \Lambda}}{\forall x B, \Pi \vdash \Lambda}\,\forall_l}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut$$

$$\Downarrow$$

$$
\begin{prooftree}
\AxiomC{$\varphi'_l\{\alpha \leftarrow t\}$}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, B\{x \leftarrow t\}$}
\AxiomC{φ_r}
\noLine
\UnaryInfC{$B\{x \leftarrow t\}, \Pi \vdash \Lambda$}
\RightLabel{cut}
\BinaryInfC{$\Gamma, \Pi \vdash \Delta, \Lambda$}
\end{prooftree}
$$

where φ'_l is obtained from φ_l by renaming all bound variables to globally fresh new ones, so that they are not equal to any free variable in the term t.

Definition 4.9 ($\rhd_{q_\exists}$)**.** Reduction of complexity of a cut-formula having an existential quantifier at its shallowest level:

$$
\begin{prooftree}
\AxiomC{φ_l}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, B\{x \leftarrow t\}$}
\RightLabel{$\exists_r$}
\UnaryInfC{$\Gamma \vdash \Delta, \exists x B$}
\AxiomC{φ_r}
\noLine
\UnaryInfC{$B\{x \leftarrow \alpha\}, \Pi \vdash \Lambda$}
\RightLabel{$\exists_l$}
\UnaryInfC{$\exists x B, \Pi \vdash \Lambda$}
\RightLabel{cut}
\BinaryInfC{$\Gamma, \Pi \vdash \Delta, \Lambda$}
\end{prooftree}
$$

$$\Downarrow$$

$$
\begin{prooftree}
\AxiomC{φ_l}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, B\{x \leftarrow t\}$}
\AxiomC{$\varphi'_r\{\alpha \leftarrow t\}$}
\noLine
\UnaryInfC{$B\{x \leftarrow t\}, \Pi \vdash \Lambda$}
\RightLabel{cut}
\BinaryInfC{$\Gamma, \Pi \vdash \Delta, \Lambda$}
\end{prooftree}
$$

where φ'_r is obtained from φ_r by renaming all bound variables to globally fresh new ones, so that they are not equal to any free variable in the term t.

Definition 4.10 ($\rhd_w$)**.** Cut-elimination over weakening inferences:

$$
\begin{prooftree}
\AxiomC{φ_l}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta$}
\RightLabel{w_r}
\UnaryInfC{$\Gamma \vdash \Delta, A$}
\AxiomC{φ_r}
\noLine
\UnaryInfC{$A, \Pi \vdash \Lambda$}
\RightLabel{cut}
\BinaryInfC{$\Gamma, \Pi \vdash \Delta, \Lambda$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{φ_l}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta, A$}
\AxiomC{φ_r}
\noLine
\UnaryInfC{$\Pi \vdash \Lambda$}
\RightLabel{w_l}
\UnaryInfC{$A, \Pi \vdash \Lambda$}
\RightLabel{cut}
\BinaryInfC{$\Gamma, \Pi \vdash \Delta, \Lambda$}
\end{prooftree}
$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$
\begin{prooftree}
\AxiomC{φ_l}
\noLine
\UnaryInfC{$\Gamma \vdash \Delta$}
\doubleLine
\RightLabel{w_r^*, w_l^*}
\UnaryInfC{$\Gamma, \Pi \vdash \Delta, \Lambda$}
\end{prooftree}
\qquad
\begin{prooftree}
\AxiomC{φ_r}
\noLine
\UnaryInfC{$\Pi \vdash \Lambda$}
\doubleLine
\RightLabel{w_r^*, w_l^*}
\UnaryInfC{$\Gamma, \Pi \vdash \Delta, \Lambda$}
\end{prooftree}
$$

Definition 4.11 ($\rhd_c$)**.** Duplication of cuts over contraction inferences[11]:

$$\dfrac{\dfrac{\overset{\varphi_l}{\Gamma \vdash \Delta, A, A}}{\Gamma \vdash \Delta, A}\,c_r \qquad \overset{\varphi_r}{A, \Pi \vdash \Lambda}}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut \qquad\qquad \dfrac{\overset{\varphi_l}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_r}{A, A, \Pi \vdash \Lambda}}{A, \Pi \vdash \Lambda}\,c_l}{\Gamma, \Pi \vdash \Delta, \Lambda}\,cut$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\dfrac{\dfrac{\dfrac{\overset{\varphi_l}{\Gamma \vdash \Delta, A, A} \qquad \overset{\varphi_r}{A, \Pi \vdash \Lambda}}{\Gamma, \Pi \vdash \Delta, \Lambda, A}\,cut \qquad \overset{\varphi_r'}{A, \Pi \vdash \Lambda}}{\Gamma, \Pi, \Pi \vdash \Delta, \Lambda, \Lambda,}\,cut}{\Gamma, \Pi \vdash \Delta, \Lambda}\,c_l^*, c_r^* \qquad\qquad \dfrac{\dfrac{\overset{\varphi_l'}{\Gamma \vdash \Delta, A} \qquad \dfrac{\overset{\varphi_l}{\Gamma \vdash \Delta, A} \qquad \overset{\varphi_r}{A, A, \Pi \vdash \Lambda}}{A, \Gamma, \Pi \vdash \Delta, \Lambda}\,cut}{\Gamma, \Gamma, \Pi \vdash \Delta, \Delta, \Lambda}\,cut}{\Gamma, \Pi \vdash \Delta, \Lambda}\,c_l^*, c_r^*$$

where φ_l' and φ_r' are variants of, respectively, φ_l and φ_r, in which the eigenvariables are renamed to preserve proof regularity.

Appendix C: Inference Swapping

This appendix defines the proof rewriting relations for inference swapping, as needed by Lemma 2.1.

Definition 4.12 (Inference Dependence)**.** An inference ρ_1 is *directly dependent* on another inference ρ_2 if and only if a main occurrence of ρ_2 is an ancestor of an auxiliary occurrence of ρ_1.

A strong quantifier inference ρ_1 is *eigenvariable-dependent* on another inference ρ_2 occurring above ρ_1 if and only if the substitution term of ρ_2 contains an occurrence of the eigenvariable of ρ_1.

An inference ρ_1 is *indirectly dependent* on another inference ρ_2 occurring above ρ_1 if and only if it is not directly dependent on ρ_2 and the auxiliary occurrences of ρ_1 have ancestors in more than one premise sequent of ρ_2.

An inference ρ_1 is *independent* of another inference ρ_2 if and only if ρ_1 is neither directly dependent nor eigenvariable-dependent nor indirectly dependent on ρ_2.

Definition 4.13 ($\gg_I$)**.** Swapping of Independent Inferences:

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta_1^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1}}{\Gamma^{\rho_1}, \Gamma_1^{\rho_2}, \Gamma_1 \vdash \Delta^{\rho_1}, \Delta_1^{\rho_2}, \Delta_1}\,\rho_1}{\Gamma^{\rho_1}, \Gamma^{\rho_2}, \Gamma_1 \vdash \Delta^{\rho_1}, \Delta^{\rho_2}, \Delta_1}\,\rho_2$$

$$\Downarrow$$

[11] The use of a purely multiplicative calculus, in which all contractions occur explicitly via contraction inferences, allows the isolation of the phenomenon of duplication of subproofs (and the need for renaming of eigenvariables) to the case of cut-reduction over contractions. Had an additive or mixed calculus been chosen, implicit contractions would occur, and its treatment would not be as transparent.

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array}}{\Gamma_1^{\rho_1},\Gamma^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta^{\rho_2},\Delta_1}\,\rho_2}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1}\,\rho_1$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array} \qquad \begin{array}{c}\varphi_2\\ \Gamma_2^{\rho_1},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2\end{array}}{\Gamma^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta_1^{\rho_2},\Delta_1,\Delta_2}\,\rho_1}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array}}{\Gamma_1^{\rho_1},\Gamma^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta^{\rho_2},\Delta_1}\,\rho_2 \qquad \begin{array}{c}\varphi_2\\ \Gamma_2^{\rho_1},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2\end{array}}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_1$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_2\\ \Gamma_2^{\rho_1},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2\end{array} \qquad \begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array}}{\Gamma^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta_1^{\rho_2},\Delta_1,\Delta_2}\,\rho_1}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\begin{array}{c}\varphi_2\\ \Gamma_2^{\rho_1},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2\end{array} \qquad \dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array}}{\Gamma_1^{\rho_1},\Gamma^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta^{\rho_2},\Delta_1}\,\rho_2}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_1$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array}}{\Gamma^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta^{\rho_1},\Delta_1^{\rho_2},\Delta_1}\,\rho_1 \qquad \begin{array}{c}\varphi_2\\ \Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_2},\Delta_2\end{array}}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array} \qquad \begin{array}{c}\varphi_2\\ \Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_2},\Delta_2\end{array}}{\Gamma_1^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta_1^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_2}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_1$$

$$\dfrac{\begin{array}{c}\varphi_2\\ \Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_2},\Delta_2\end{array} \qquad \dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{array}}{\Gamma^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta^{\rho_1},\Delta_1^{\rho_2},\Delta_1}\,\rho_1}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_2}{\Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_2},\Delta_2} \qquad \overset{\varphi_1}{\Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1}}{\Gamma_1^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta_1^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_2}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_1$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1} \qquad \overset{\varphi_2}{\Gamma_2^{\rho_1},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2}}{\Gamma^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta_1^{\rho_2},\Delta_1,\Delta_2}\,\rho_1 \qquad \overset{\varphi_3}{\Gamma_3^{\rho_2},\Gamma_3 \vdash \Delta_3^{\rho_2},\Delta_3}}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2,\Gamma_3 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2,\Delta_3}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1} \qquad \overset{\varphi_3}{\Gamma_3^{\rho_2},\Gamma_3 \vdash \Delta_3^{\rho_2},\Delta_3}}{\Gamma_1^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_3 \vdash \Delta_1^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_3}\,\rho_2 \qquad \overset{\varphi_2}{\Gamma_2^{\rho_1},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2}}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2,\Gamma_3 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2,\Delta_3}\,\rho_1$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^{\rho_1},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1} \qquad \overset{\varphi_2}{\Gamma_2^{\rho_1},\Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2^{\rho_2},\Delta_2}}{\Gamma^{\rho_1},\Gamma_2^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta_2^{\rho_2},\Delta_1,\Delta_2}\,\rho_1 \qquad \overset{\varphi_3}{\Gamma_3^{\rho_2},\Gamma_3 \vdash \Delta_3^{\rho_2},\Delta_3}}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2,\Gamma_3 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2,\Delta_3}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\overset{\varphi_1}{\Gamma_1^{\rho_1},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1} \qquad \dfrac{\overset{\varphi_2}{\Gamma_2^{\rho_1},\Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2^{\rho_2},\Delta_2} \qquad \overset{\varphi_3}{\Gamma_3^{\rho_2},\Gamma_3 \vdash \Delta_3^{\rho_2},\Delta_3}}{\Gamma_2^{\rho_1},\Gamma^{\rho_2},\Gamma_2,\Gamma_3 \vdash \Delta_2^{\rho_1},\Delta^{\rho_2},\Delta_2,\Delta_3}\,\rho_2}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2,\Gamma_3 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2,\Delta_3}\,\rho_1$$

$$\dfrac{\overset{\varphi_3}{\Gamma_3^{\rho_2},\Gamma_3 \vdash \Delta_3^{\rho_2},\Delta_3} \qquad \dfrac{\overset{\varphi_1}{\Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1} \qquad \overset{\varphi_2}{\Gamma_2^{\rho_1},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2}}{\Gamma^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta_1^{\rho_2},\Delta_1,\Delta_2}\,\rho_1}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2,\Gamma_3 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2,\Delta_3}\,\rho_2$$

$$\Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_3}{\Gamma_3^{\rho_2},\Gamma_3 \vdash \Delta_3^{\rho_2},\Delta_3} \qquad \overset{\varphi_1}{\Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1}}{\Gamma_1^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_3 \vdash \Delta_1^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_3}\,\rho_2 \qquad \overset{\varphi_2}{\Gamma_2^{\rho_1},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2}}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2,\Gamma_3 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2,\Delta_3}\,\rho_1$$

$$
\dfrac{\begin{matrix}\varphi_3\\ \Gamma_3^{\rho_2},\Gamma_3 \vdash \Delta_3^{\rho_2},\Delta_3\end{matrix} \qquad \dfrac{\begin{matrix}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1\end{matrix} \qquad \begin{matrix}\varphi_2\\ \Gamma_2^{\rho_1},\Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2^{\rho_2},\Delta_2\end{matrix}}{\Gamma^{\rho_1},\Gamma_2^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta_2^{\rho_2},\Delta_1,\Delta_2}\,\rho_1}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2,\Gamma_3 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2,\Delta_3}\,\rho_2
$$

$$\Downarrow$$

$$
\dfrac{\begin{matrix}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1\end{matrix} \qquad \dfrac{\begin{matrix}\varphi_3\\ \Gamma_3^{\rho_2},\Gamma_3 \vdash \Delta_3^{\rho_2},\Delta_3\end{matrix} \qquad \begin{matrix}\varphi_2\\ \Gamma_2^{\rho_1},\Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2^{\rho_2},\Delta_2\end{matrix}}{\Gamma_2^{\rho_1},\Gamma^{\rho_2},\Gamma_2,\Gamma_3 \vdash \Delta_2^{\rho_1},\Delta^{\rho_2},\Delta_2,\Delta_3}\,\rho_2}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2,\Gamma_3 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2,\Delta_3}\,\rho_1
$$

Definition 4.14 ($\gg_{ID}$)**.** Distributional Swapping of Indirectly Dependent Inferences:

$$
\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{matrix} \qquad \begin{matrix}\varphi_2\\ \Gamma_2^{\rho_1},\Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2^{\rho_2},\Delta_2\end{matrix}}{\Gamma^{\rho_1},\Gamma_1^{\rho_2},\Gamma_2^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta_1^{\rho_2},\Gamma_2^{\rho_2},\Delta_1,\Delta_2}\,\rho_1}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_2
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_1\end{matrix}}{\Gamma_1^{\rho_1},\Gamma_1^{\rho_2},\Gamma_2^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta_1^{\rho_2},\Delta_2^{\rho_2}\Delta_1}\,w^*}{\Gamma_1^{\rho_1},\Gamma^{\rho_2},\Gamma_1 \vdash \Delta_1^{\rho_1},\Delta^{\rho_2},\Delta_1}\,\rho_2 \qquad \dfrac{\dfrac{\begin{matrix}\varphi_2\\ \Gamma_2^{\rho_1},\Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_2^{\rho_2},\Delta_2\end{matrix}}{\Gamma_2^{\rho_1},\Gamma_1^{\rho_2},\Gamma_2^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta_1^{\rho_2},\Delta_2^{\rho_2},\Delta_2}\,w^*}{\Gamma_2^{\rho_1},\Gamma^{\rho_2},\Gamma_2 \vdash \Delta_2^{\rho_1},\Delta^{\rho_2},\Delta_2}\,\rho_2}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta^{\rho_2},\Delta_1,\Delta_2}\,\rho_1}{\Gamma^{\rho_1},\Gamma^{\rho_2},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho_1},\Delta^{\rho_2},\Delta_1,\Delta_2}\,c^*
$$

Definition 4.15 ($\gg_{IDC}$)**.** Swapping of indirectly dependent contractions:

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma_1,\Gamma_\rho,\Gamma_\rho \vdash \Delta_1,\Delta_\rho,\Delta_\rho\end{matrix}}{\Gamma_1,\Gamma_\rho,\Pi_\rho \vdash \Delta_1,\Delta_\rho,\Lambda_\rho}\,\rho}{\Gamma_1,\Pi_\rho,\Pi_\rho \vdash \Delta_1,\Lambda_\rho,\Lambda_\rho}\,\rho}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,c^*
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma_1,\Gamma_\rho,\Gamma_\rho \vdash \Delta_1,\Delta_\rho,\Delta_\rho\end{matrix}}{\Gamma_1,\Gamma_\rho \vdash \Delta_1,\Delta_\rho}\,c^*}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,\rho
$$

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\varphi_1\\ \Gamma_1,\Gamma_1^\rho,\Gamma_1^\rho \vdash \Delta_1,\Delta_\rho,\Delta_\rho\end{matrix} \qquad \begin{matrix}\varphi_2\\ \Gamma_2,\Gamma_2^\rho \vdash \Delta_2,\Delta_2^\rho\end{matrix}}{\Gamma_1,\Gamma_\rho,\Pi_\rho \vdash \Delta_1,\Delta_\rho,\Lambda_\rho}\,\rho \qquad \begin{matrix}\varphi_2\\ \Gamma_2,\Gamma_2^\rho \vdash \Delta_2,\Delta_2^\rho\end{matrix}}{\Gamma_1,\Pi_\rho,\Pi_\rho \vdash \Delta_1,\Lambda_\rho,\Lambda_\rho}\,\rho}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,c^*
$$

$$\Downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1,\Gamma_1^\rho,\Gamma_1^\rho \vdash \Delta_1,\Delta_1^\rho,\Delta_1^\rho\end{array}}{\Gamma_1,\Gamma_\rho \vdash \Delta_1,\Delta_\rho}\,c^* \qquad \begin{array}{c}\varphi_2\\ \Gamma_2,\Gamma_2^\rho \vdash \Delta_2,\Delta_2^\rho\end{array}}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,\rho$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_2\\ \Gamma_2,\Gamma_2^\rho \vdash \Delta_2,\Delta_2^\rho\end{array} \qquad \dfrac{\begin{array}{c}\varphi_2\\ \Gamma_2,\Gamma_2^\rho \vdash \Delta_2,\Delta_2^\rho\end{array} \qquad \begin{array}{c}\varphi_1\\ \Gamma_1,\Gamma_1^\rho,\Gamma_1^\rho \vdash \Delta_1,\Delta_\rho,\Delta_\rho\end{array}}{\Gamma_1,\Gamma_\rho,\Pi_\rho \vdash \Delta_1,\Delta_\rho,\Lambda_\rho}\,\rho}{\Gamma_1,\Pi_\rho,\Pi_\rho \vdash \Delta_1,\Lambda_\rho,\Lambda_\rho}\,\rho}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,c^*$$

$$\Downarrow$$

$$\dfrac{\begin{array}{c}\varphi_2\\ \Gamma_2,\Gamma_2^\rho \vdash \Delta_2,\Delta_2^\rho\end{array} \qquad \dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1,\Gamma_1^\rho,\Gamma_1^\rho \vdash \Delta_1,\Delta_1^\rho,\Delta_1^\rho\end{array}}{\Gamma_1,\Gamma_\rho \vdash \Delta_1,\Delta_\rho}\,c^*}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,\rho$$

Definition 4.16 ($\gg_C$)**.** Distributional Swapping over contractions:

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1,\Gamma_\rho,\Gamma'_\rho \vdash \Delta_1,\Delta_\rho,\Delta'_\rho\end{array}}{\Gamma_1,\Gamma_\rho \vdash \Delta_1,\Delta_\rho}\,c^*}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,\rho$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1,\Gamma_\rho,\Gamma'_\rho \vdash \Delta_1,\Delta_\rho,\Delta'_\rho\end{array}}{\Gamma_1,\Gamma_\rho,\Gamma_\rho \vdash \Delta_1,\Delta_\rho,\Delta_\rho}\,w^*}{\Gamma_1,\Gamma_\rho,\Pi_\rho \vdash \Delta_1,\Delta_\rho,\Lambda_\rho}\,\rho}{\Gamma_1,\Pi_\rho,\Pi_\rho \vdash \Delta_1,\Lambda_\rho,\Lambda_\rho}\,\rho}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,c^*$$

$$\dfrac{\dfrac{\begin{array}{c}\varphi_1\\ \Gamma_1,\Gamma_1^\rho,\Gamma_1\rho' \vdash \Delta_1,\Delta_1^\rho,\Delta_1^{\rho'}\end{array}}{\Gamma_1,\Gamma_\rho \vdash \Delta_1,\Delta_\rho}\,c^* \qquad \begin{array}{c}\varphi_2\\ \Gamma_2,\Gamma_2^\rho \vdash \Delta_2,\Delta_2^\rho\end{array}}{\Gamma_1,\Pi_\rho \vdash \Delta_1,\Lambda_\rho}\,\rho$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1,\Gamma^{\rho}_1,\Gamma^{\rho'}_1 \vdash \Delta_1,\Delta^{\rho}_1,\Delta^{\rho'}_1}}{\Gamma_1,\Gamma^{\rho}_1,\Gamma^{\rho}_1 \vdash \Delta_1,\Delta_{\rho},\Delta_{\rho}}\,w^* \qquad \overset{\varphi_2}{\Gamma_2,\Gamma^{\rho}_2 \vdash \Delta_2,\Delta^{\rho}_2}}{\Gamma_1,\Gamma_{\rho},\Pi_{\rho} \vdash \Delta_1,\Delta_{\rho},\Lambda_{\rho}}\,\rho \qquad \overset{\varphi_2}{\Gamma_2,\Gamma^{\rho}_2 \vdash \Delta_2,\Delta^{\rho}_2}}{\Gamma_1,\Pi_{\rho},\Pi_{\rho} \vdash \Delta_1,\Lambda_{\rho},\Lambda_{\rho}}\,\rho}{\Gamma_1,\Pi_{\rho} \vdash \Delta_1,\Lambda_{\rho}}\,c^*
$$

$$
\dfrac{\overset{\varphi_2}{\Gamma_2,\Gamma^{\rho}_2 \vdash \Delta_2,\Delta^{\rho}_2} \qquad \dfrac{\overset{\varphi_1}{\Gamma_1,\Gamma^{\rho}_1,\Gamma^{\rho'}_1 \vdash \Delta_1,\Delta^{\rho}_1,\Delta^{\rho'}_1}}{\Gamma_1,\Gamma_{\rho} \vdash \Delta_1,\Delta_{\rho}}\,c^*}{\Gamma_1,\Pi_{\rho} \vdash \Delta_1,\Lambda_{\rho}}\,\rho
$$

$$\Downarrow$$

$$
\dfrac{\dfrac{\overset{\varphi_2}{\Gamma_2,\Gamma^{\rho}_2 \vdash \Delta_2,\Delta^{\rho}_2} \qquad \dfrac{\overset{\varphi_2}{\Gamma_2,\Gamma^{\rho}_2 \vdash \Delta_2,\Delta^{\rho}_2} \qquad \dfrac{\overset{\varphi_1}{\Gamma_1,\Gamma^{\rho}_1,\Gamma^{\rho'}_1 \vdash \Delta_1,\Delta^{\rho}_1,\Delta^{\rho'}_1}}{\Gamma_1,\Gamma^{\rho}_1,\Gamma^{\rho}_1 \vdash \Delta_1,\Delta_{\rho},\Delta_{\rho}}\,w^*}{\Gamma_1,\Gamma_{\rho},\Pi_{\rho} \vdash \Delta_1,\Delta_{\rho},\Lambda_{\rho}}\,\rho}{\Gamma_1,\Pi_{\rho},\Pi_{\rho} \vdash \Delta_1,\Lambda_{\rho},\Lambda_{\rho}}\,\rho}{\Gamma_1,\Pi_{\rho} \vdash \Delta_1,\Lambda_{\rho}}\,c^*
$$

Definition 4.17 ($\gg_{WI}$)**.** Downward swapping of weakening inferences over independent inferences.

$$
\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^{\rho}_1,\Gamma \vdash \Delta^{\rho}_1,\Delta}}{F,\Gamma^{\rho}_1,\Gamma \vdash \Delta^{\rho}_1,\Delta}\,w_l}{F,\Gamma^{\rho},\Gamma \vdash \Delta^{\rho},\Delta}\,\rho
\qquad\qquad
\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^{\rho}_1,\Gamma \vdash \Delta^{\rho}_1,\Delta}}{\Gamma^{\rho}_1,\Gamma \vdash \Delta^{\rho}_1,\Delta,F}\,w_r}{F,\Gamma^{\rho},\Gamma \vdash \Delta^{\rho},\Delta,F}\,\rho
$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$
\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^{\rho}_1,\Gamma \vdash \Delta^{\rho}_1,\Delta}}{\Gamma^{\rho},\Gamma \vdash \Delta^{\rho},\Delta}\,\rho}{F,\Gamma^{\rho},\Gamma \vdash \Delta^{\rho},\Delta}\,w_l
\qquad\qquad
\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^{\rho}_1,\Gamma \vdash \Delta^{\rho}_1,\Delta}}{\Gamma^{\rho},\Gamma \vdash \Delta^{\rho},\Delta}\,\rho}{\Gamma^{\rho},\Gamma \vdash \Delta^{\rho},\Delta,F}\,w_l
$$

$$
\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^{\rho}_1,\Gamma_1 \vdash \Delta^{\rho}_1,\Delta_1}}{F,\Gamma^{\rho}_1,\Gamma_1 \vdash \Delta^{\rho}_1,\Delta_1}\,w_l \qquad \overset{\varphi_2}{\Gamma^{\rho}_2,\Gamma_2 \vdash \Delta^{\rho}_2,\Delta_2}}{F,\Gamma^{\rho},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho},\Delta_1,\Delta_2}\,\rho
\qquad\qquad
\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^{\rho}_1,\Gamma_1 \vdash \Delta^{\rho}_1,\Delta_1} \qquad \overset{\varphi_2}{\Gamma^{\rho}_2,\Gamma_2 \vdash \Delta^{\rho}_2,\Delta_2}}{\Gamma^{\rho},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho},\Delta_1,\Delta_2}\,\rho}{F,\Gamma^{\rho},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho},\Delta_1,\Delta_2}\,w_l
$$

$$
\dfrac{\dfrac{\overset{\varphi_1}{\Gamma^{\rho}_1,\Gamma_1 \vdash \Delta^{\rho}_1,\Delta_1}}{\Gamma^{\rho}_1,\Gamma_1 \vdash \Delta^{\rho}_1,\Delta_1,F}\,w_r \qquad \overset{\varphi_2}{\Gamma^{\rho}_2,\Gamma_2 \vdash \Delta^{\rho}_2,\Delta_2}}{\Gamma^{\rho},\Gamma_1,\Gamma_2 \vdash \Delta^{\rho},\Delta_1,F,\Delta_2}\,\rho
$$

$$\Downarrow$$

$$\Downarrow \qquad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1} \qquad \overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\,\rho}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, F, \Delta_2}\,w_r$$

$$\dfrac{\overset{\varphi_1}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1} \qquad \dfrac{\overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{F, \Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}\,w_l}{F, \Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\,\rho \qquad \dfrac{\overset{\varphi_1}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1} \qquad \dfrac{\overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2, F}\,w_r}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, F, \Delta_2}\,\rho$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1} \qquad \overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\,\rho}{F, \Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\,w_l \qquad \dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1} \qquad \overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\,\rho}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, F, \Delta_2}\,w_r$$

Definition 4.18 ($\gg_{WD}$)**.** Downward swapping of weakening inferences over directly dependent inferences.

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta}}{\Gamma_1^\rho, \Gamma \vdash \Delta_1^\rho, \Delta}\,w^*}{\Gamma^\rho, \Gamma \vdash \Delta^\rho, \Delta}\,\rho$$

$$\Downarrow$$

$$\dfrac{\overset{\varphi_1}{\Gamma \vdash \Delta}}{\Gamma^\rho, \Gamma \vdash \Delta^\rho, \Delta}\,w^*$$

$$\dfrac{\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1}}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1}\,w^* \qquad \overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2}}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\,\rho \qquad \dfrac{\overset{\varphi_2}{\Gamma_2^\rho, \Gamma_2 \vdash \Delta_2^\rho, \Delta_2} \qquad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1}}{\Gamma_1^\rho, \Gamma_1 \vdash \Delta_1^\rho, \Delta_1}\,w^*}{\Gamma^\rho, \Gamma_2, \Gamma_1 \vdash \Delta^\rho, \Delta_2, \Delta_1}\,\rho$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1}}{\Gamma^\rho, \Gamma_1, \Gamma_2 \vdash \Delta^\rho, \Delta_1, \Delta_2}\,w^* \qquad \dfrac{\overset{\varphi_1}{\Gamma_1 \vdash \Delta_1}}{\Gamma^\rho, \Gamma_2, \Gamma_1 \vdash \Delta^\rho, \Delta_2, \Delta_1}\,w^*$$

Definition 4.19 ($\gg_W$)**.** The proof rewriting relation for *downward swapping of weakening* is:

$$\gg_W \doteq (\gg_{WI} \cup \gg_{WD})$$

Definition 4.20 ($\gg$)**.** The proof rewriting relation for *inference swapping* is:

$$\gg \doteq (\gg_I \cup \gg_{ID} \cup \gg_{IDC} \cup \gg_C \cup \gg_W)$$